Pollution, Mortality and Ramsey Taxes*

Aditya Goenka†, Saqib Jafarey‡, William Pouliot§

February 16, 2017

Abstract: We study Ramsey (second-best) optimal policy in an overlapping generations economy where pollution increases mortality. Economic activity causes pollution: it has a negative effect on life expectancy while higher income has a prophylactic effect. These counteracting effects can make the growth-survival relationship non-concave and cause multiple steady states and a poverty trap. An increase in exogenous abatement taxes can increase the basin of the poverty trap. We study dynamically consistent second-best abatement taxes where the planner takes the optimal savings function of successive generations of agents as given in choosing the taxes. The optimal tax is a non-homogeneous and increasing function of the contemporaneous capital stock. The response of the capital stock to the optimal tax can make abatement policy an independent source of non-monotonicity that leads to non-existence and multiplicity of steady states and introduces qualitative changes in local dynamics around steady states, such as oscillations when there were none.

Keywords: Overlapping generations model, pollution, Ramsey taxes, mortality, poverty traps, endogenous fluctuations, optimal environmental policy.

JEL Classification: O11, O13, O23, O44, E32, H21, H23.

---

*We thank seminar audiences at CEF 2010, SCE Prague 2012, SWET Paris 2012, ISI Development Conference 2012, Asian Econometric Society Meetings 2012, GE Days York 2013, Warwick Summer School on Economic Growth, 2014, Future of GE Conference (NYU-AUD) 2015, EWGET Glasgow 2016, Universitat Autonoma Barcelona, Birmingham, City University, Kyung Hee University, Liverpool, NUS, Rochester, and especially Alioso Araujo, Joydeep Bhattacharya, Shankha Chakraborty, Christian Ghiglino, Piero Gottardi, Cuong Le Van, Lin Liu, Michael Magill, Stephen Morris, Hyun Park, Apostolis Philippopoulos, Martine Quinzii, Thomas Seegmuller, Karl Shell, Gilles Saint-Paul, Mich Tvede, Yiannis Vailakis, Thierry Verdier, and Randy Wright for their helpful comments. Jafarey and Goenka were partly funded by British Academy Small Scale Research Grant (SG-47102), and Goenka also by FASS Research Grant R-122-000-131-112. The usual disclaimer applies.

†Department of Economics, University of Birmingham, JG Smith Building, Birmingham, UK B15 2TT; Email: a.goenka@bham.ac.uk

‡Department of Economics, City University London, Northampton Square, London EC1V 0HB; Email: s.s.jafarey@city.ac.uk

§Department of Economics. University of Birmingham, JG Smith Building, Birmingham, UK B15 2TT; Email: w.pouliot@bham.ac.uk

1 Introduction

Pollution, especially particulate matter (PM) but also nitrogen dioxide, sulphur dioxide, and ozone, leads to increase of cardiovascular and respiratory diseases and causes premature mortality.[1] This paper studies optimal abatement policy in an overlapping generations model that incorporates a pollution externality on premature mortality. There is a three way link between pollution, mortality and economic growth: while economic growth reduces mortality rates through the effect of higher income and better health outcomes, it also generates pollution which increases them.[2] Changes in mortality in turn affect savings decisions and thus economic growth and thereby, growth in the rate of pollution flows.

Recent economic literature has recognized the possibility that multiple steady states, poverty traps and cycles can arise from the interplay between the three factors and proposed various policy options, via either golden-rule, steady state analysis or Pigouvian taxes, to offset these outcomes.[3] It has, however, not studied Ramsey taxation, where a planner chooses an optimal sequence of state-contingent policy actions funded by a sequence of distortionary taxes, taking the optimal choices of private agents as given. This viewpoint is important as it addresses the issues of dynamic consistency and implementability which are both problematic in overlapping generations. Looking at state contingent taxes is also important to understand the transition dynamics.

We study a two-period overlapping generations model where the probability of survival into old age is determined endogenously (Chakraborty [2004], and Bhattacharya and Qiao [2007]). Production of a single consumption-capital good creates pollution as a by-

[1] Water pollution, carcinogens of both gaseous and soil contaminant types, heavy metals (such as mercury), persistent organic pollutants (POPs such as DDT, dioxin), etc. are other types of pollution that lead to premature mortality. There is robust micro evidence that exposure to particulate matter PM_{10} and $PM_{2.5}$, leads to increased cardiovascular disease, chronic obstructive pulmonary disease (COPD) and, controlling for other factors, an increase in mortality (see Ayres [2006], Huang *et al.* [2012], Evans *et al.* [2013], Miller *et al.* [2007], Pope *et al.* [2004], HEI [2010], Viegi *et al.* [2006]). A 10 μg per cubic meter increase in PM_{10} leads to an increase in mortality by 0.51% and if other gases such as ozone, nitrogen dioxide, sulfur dioxide and carbon monoxide which are correlated with PM_{10} are taken into account the distribution of mortality shifts to the right (Samet *et al.* [2000]). These effects are present in both developed and developing countries.

[2] Wang, Zhang, and Bhattarcharya [2015] study a complementary model where pollution affects morbidity but not mortality.

[3] See Jouvet *et al.* [2010]; Mariani *et al.* [2010], Varvarigos [2008], [2014]; Palivos and Varvarigos [2010] and Raffin and Seegmuller [2014]. Also see Stokey [1998] who studies the first best problem in a long-lived agent model with environmental externalities but no mortality effects. For earlier studies of taxes relying on steady state analysis to correct environmental externalities in overlapping generations models see Bovenberg and Heijdra [1998], John and Pecchenino [1994], and John *et al.* [1995]

product. Increased pollution increases the probability of premature death but increased income has a prophylactic effect on mortality.[4] The positive effect of income on mortality follows the literature which has pointed out that increased income can counteract some of the adverse effects of pollution via better nutrition and greater access to health care.[5]

We study the dynamic, competitive equilibria of the economy. The two contrary forces that affect mortality can result in a non-convexity that gives rise to poverty traps and differing effects of environmental policy between rich and poor countries. As a benchmark, we first look at the impact of an exogenous linear income (wage) tax whose proceeds are used for pollution abatement. Under an exogenous tax, there can be a low capital, poverty trap, steady state and in which there is lower per capita consumption and life expectancy and a high capital steady state in which per capita consumption and life expectancy are both higher. We refer to the high capital steady state as neoclassical because its essential features are those of the steady state in a neoclassical growth model. The poverty trap is a source: any path that starts with a lower capital stock converges over time to a zero-consumption or trivial steady state. Furthermore, increases in the uniform tax can increase the steady state capital in the neoclassical steady state while simultaneously widening the basin of attraction of the trivial steady state.[6]

The main contribution of the paper is in characterising the optimal abatement policy, financed by distortionary taxes. There is a well-known commitment problem in imposing taxes on future generations (see Ghiglino and Tvede [2000], and the survey by Erosa and Gervais [2001]). Thus, we assume that the tax is set in a dynamically consistent fashion, *i.e.* as a function of that period's capital stock. Such a state-dependent policy requires no pre-commitment.[7]

We assume that the government imposes a wage tax on the young in each period. This tax affects the young's savings behaviour. Private savings determine next period's capital stock which imposes contradictory externalities on the next generation: a higher capital stock means higher incomes which reduce the next generation's mortality but also means

[4]Chakraborty [2004] assumes only the second positive effect, i.e. survival depends on the stock of health which is an exogenous linear function of wage income.

[5]Preston [1975] was one of the earliest papers to document the positive effect of income on life expectancy. The recent survey by Cutler *et al.* [2006] documents this effect across countries and within countries. In their interpretation, income alone is not sufficient but it is correlated with willingness for effective public health delivery.

[6]The former possibility is known: environmental degradation imposes costs that are external to each decision-maker so any policy that offsets this externality helps reduce these costs and if the balance is right, actually promotes growth (see Economides and Philippopoulos [2008], John *et al.* [1995], Mariani *et al.* [2010], and Palivos and Varvarigos [2010] or an analysis of such effects in a variety of settings).

[7]John *et al.* [1995] highlight the problem of using pre-committed Pigouvian taxes in such an environment.

higher emissions which increase it. It is not possible in our model to offset the externality entirely by means of the wage tax.[8] Thus, the government has only a second-best instrument to maximise a weighted sum of life-cycle utilities of all generations, subject to each generation's incentive constraints regarding savings behaviour.

We establish the existence of the optimal tax function with the following characteristics. First, below a threshold level of capital, the optimal tax is zero and there is no pollution abatement, as at low levels of pollution, the marginal effects of additional pollution are negligible.[9] Second, in the region of positive taxation, the optimal tax is weakly increasing and concave in the capital stock. Third, the optimal tax at a given capital stock increases with the size of the inter-generational discount factor of the planner.

We show that the interaction of the optimal tax policy with optimal savings leads to new dynamic phenomena. First, when the underlying economy with exogenous taxes admits two steady states, optimal taxes can reverse their stability properties: a poverty trap can act as a sink while a neoclassical steady state acts as a source. Second, optimal taxes may introduce endogenous fluctuations in the neighbourhood of either steady state. Third, optimal taxes may affect the very problem of existence and uniqueness of steady states. These dynamics arise even when the government places relatively high weight on the utility of future generations.

With state-contingent taxation, the interaction of optimal taxes with savings decisions exerts its own dynamic effect. When the tax rate is fixed, a stationary savings function drives the entire path of capital accumulation. With optimal taxation, a stationary tax policy *function* replaces a fixed tax and the private savings functions shifts with each change in the tax rate. Now the capital stock evolves along a locus of shifting saving functions. This adds an additional dimension to the local dynamics and this is what leads to qualitative changes in dynamic behaviour.

The paper also relates to a broader literature that addresses the interaction of economic policies and endogenous fluctuations in dynamic general equilibrium (see Woodford [1994a]). One strand of this literature (see Goenka and Liu [2012] and Grandmont [1985]) shows that state-dependent economic policies can be used to stabilize endogenous economic fluctuations. Another strand shows that simple, non-state dependent feedback

[8] We discuss later why the wage tax is the only reasonable tax instrument in our framework.

[9] Palivos and Varvarigos [2010] derive a similar result for a policy of maximising survival probability rather than intergenerational welfare. Note that the latter does not increase unambiguously with survival probability as the direct positive effect of survival on generational welfare and the indirect positive effect that arises due to higher incentives to accumulate capital at given interest rates can be offset by the indirect negative effect that higher survival rates have on the net return to savings. Our welfare criterion captures all three effects.

policies can themselves be a source of endogenous economic fluctuations (see Goenka [1994a], Goenka [1994b], Grandmont [1986], Smith [1994], Woodford [1994b]), while state-dependent feedback policies may eliminate these. In this paper we present a different type of difficulty: when the private response to optimal policy shows potential non-convexities and the policy-maker is restricted to second-best instruments, state-dependent policies can generate non-linear dynamics in the evolution of state variables.

The plan of the paper is as follows. In section 2, the benchmark model is developed. Section 3 studies the effects of exogenous (constant) taxes, and section 4 studies the second-best optimal tax. In this section we first characterize properties of the optimal tax function, and then study the dynamics of the equilibrium trajectories. The final section concludes.

2 Model

Time is discrete and denoted by $t = 0, 1, \ldots$. Each period a new generation is born, indexed by its period of birth. A generation consists of a continuum of agents normalized to measure one. Agents born in period t live at most until the end of period $t+1$. There is uncertainty as to whether an individual will survive till old age. The probability that an agent born in period t lives until the end of period $t+1$ is denoted by π_t, while with probability $1 - \pi_t$ the agent dies at the end of period t.

Each agent supplies one unit of labour inelastically when young and receives a wage w_t which is used to finance current consumption, c_t^y and savings for old age, s_t. Old agents have no labour endowment and live entirely off the proceeds of their savings. Following the literature on uncertain lifetimes, we assume that there is a perfect annuity market in which young agents buy annuities from perfectly competitive intermediaries who lend out the proceeds to firms for investment in productive capital. Each unit of time t investment results in one unit of time $t+1$ capital, k_{t+1} which becomes immediately available for production and fully depreciates in that period. Thus,

$$k_{t+1} = s_t \tag{1}$$

At time $t = 0$, k_0 is exogenously given.

2.1 Production and factor prices

The production function is Cobb-Douglas and displays constant returns to scale. It can be expressed in intensive form:

$$y_t = Ak_t^{\alpha};$$

where y is output per worker and k is capital per worker.

The gross returns to capital and labour r_t and w_t respectively, are equal to their marginal products:

$$w_t = (1-\alpha)Ak_t^{\alpha}; \quad (2)$$

$$r_t = \frac{\alpha A}{k_t^{1-\alpha}}. \quad (3)$$

As a positive fraction of savers do not live into old age, the return on period t savings for those who survive is r_{t+1}/π_t.

2.2 Pollution emission and abatement

The production of final output creates a proportionate flow of pollutants $z_t = \gamma y_t$. Note that the relevant measure of pollution is proportional to the gross, not the per-capita, rate of output. However, because population is normalised to unity, per-capita and gross quantities are numerically identical so for notational consistency we use use lower case z to denote pollution flows and relate it to y. For indeterminate population sizes, the constant γ could be interpreted as the product of a technological rate of emissions and a scale multiplier which converts per-capita output into gross output . Furthermore, the type of pollution we are modelling here consists of PM_{10} and similar particulate matter and pollutants such as NO_x which have been linked to to health effects. The evidence shows that such pollutants are short-lived, except in certain areas characterised by their geography and the nature of economic activity, so that they can be treated as a flow (Varotsos *et al.* [2005], Windsor and Toumi [2001], Zeka *et al.* [2005]). This is different from the issue of greenhouse gas build-up that arises in the global warming literature.[10]

Environmental policy is implemented by a planner that imposes an environmental tax, τ_t on the wage incomes of the contemporaneous young.[11] The proceeds are spent on

[10]In earlier versions of the paper, Goenka *et. al.* [2012]) we show that allowing for persistence of pollution does not affect results under some conditions.

[11]The reason for restricting the incidence of environmental taxes to the young generation is explained in the section where the optimal tax policy is derived.

operating a clean-up technology that reduces the flow of pollutants. We assume that this technology can only be operated by a central authority so that individual agents do not have the means to abate privately.[12]

The efficiency of this technology is denoted by $\chi > 0$, and the reduction in pollution flows, is assumed to be a linear function of tax-financed expenditures. Thus the net flow of pollutants is:

$$z_t = \gamma y_t - \chi \tau_t w_t;$$

which, after substituting for w_t and redefining terms, simplifies to

$$z_t = \gamma(1 - \psi\tau_t)Ak_t^{\alpha}. \tag{4}$$

where $\psi = \chi(1-\alpha)/\gamma$. We assume $\psi \in [0,1]$ to avoid the possibility that as a result of abatement, the flow of pollution is negative.[13]

2.3 Probability of survival and the rationale for environmental policy

We assume that the probability of survival into old age is identical for all agents and is represented by a twice continuously differentiable function of y_t and z_t. Longevity is increasing in per-capita income and decreasing in pollution. If per-capita income is zero, the survival probability is at some minimal level regardless of the stock of pollution and as the stock of pollution approaches infinity, survival probability tends to zero regardless of the level of income.

Assumption 1

$$\begin{aligned}
\pi_t = \pi(k_t) &= \pi(y(k_t), z(k_t)); && (5)\\
\pi &\in [0,1], \;\; \forall y \geq 0 \;\&\; \forall z \geq 0; && (6)\\
\frac{\partial \pi}{\partial y} \equiv \pi_y(y,z) &\geq 0, \;\; \forall y \geq 0; && (7)\\
\frac{\partial \pi}{\partial z} \equiv \pi_z(y,z) &\leq 0, \;\; \forall z \geq 0; && (8)\\
\pi(0,z) &= \underline{\pi} \in [0,1], \; \forall z \geq 0; && (9)\\
\pi(y,\infty) &= 0, \; \forall y \geq 0. && (10)
\end{aligned}$$

[12]Some papers have considered private abatement in contexts in which the benefits of pollution abatement are unambiguously positive, see John and Pecchenino [1994], John *et. al.* [1995], and Mariani *et. al.* [2010]. In our model this is not the case, see Sections 4.1 and 4.2.

[13]We can dispense with this assumption but an interior steady state may not exist. In addition we have to assume that ψ is small enough to ensure that second order conditions for the optimal tax policy function to hold, see Lemma 2.

The only consequence of pollution in this model is that it creates a negative external effect on expected lifetimes.[14] Given the overlapping generations framework this externality affects the young generation alone by affecting their expected lifetime utility. As only the young work, the output is not affected by pollution directly. Thus, there is a potential for welfare improvement by means of a tax on the young, the proceeds of which are spent on abating pollution. Future generations are affected indirectly through the effects of pollution on savings of the current generation, i.e. the next period's capital stock.

2.4 Preferences

In order to derive closed form solutions we assume that each agent born at time t has a time-separable expected utility function, U over consumption when young c^y and when old c^o:[15]

$$U^t = \ln c_t^y + \pi_t \ln c_{t+1}^o;$$

which the agent maximises subject to the life-cycle budget constraints:

$$c_t^y \leq (1-\tau_t)w_t - s_t; \tag{11}$$

$$c_{t+1}^o \leq \frac{r_{t+1}}{\pi_t} s_t; \tag{12}$$

where s_t is the young agent's savings and c_{t+1}^o is *ex post* consumption for an agent who survives into old-age.

Taking the first-order condition with respect to savings,

$$-\frac{1}{c_t^y} + \frac{\pi_t}{c_{t+1}^o}\frac{r_{t+1}}{\pi_t} = 0;$$

and combining with equations (11), (12) and (3), results in the following equation:

$$s_t = \frac{\pi_t}{1+\pi_t} A \cdot (1-\tau_t)(1-\alpha)k_t^\alpha.$$

2.5 Equilibrium

Using the market clearing condition, *i.e.* substituting into equation (1) we have:

$$k_{t+1} = \frac{\pi_t}{1+\pi_t} A \cdot (1-\tau_t)(1-\alpha)k_t^\alpha. \tag{13}$$

The path of the capital stock is traced out by recursive application of equation (13) from a given k_0 while the accompanying evolution of the flow of pollution follows from recursively applying equation (4). The other variables are updated similarly.

[14] Wang, Zhang, and Bhattacharya (2015) study a complementary model where pollution affects morbidity and not mortality and focus on issues of insurance.

[15] The qualitative results hold under more general utility functions.

3 Exogenous Taxes

To understand the benchmark case, we first consider the case of exogenous taxes, τ. We examine the dynamics in the model and the effects of varying the tax rate which helps in the characterization of the optimal policy.

3.1 Dynamics

A steady state is described by the following equations:

$$\pi = \pi(k) = \pi(y(k), z(k)); \tag{14}$$

$$k = \frac{\pi(k)}{1+\pi(k)} A \cdot (1-\tau)(1-\alpha)k^{\alpha}; \tag{15}$$

$$z = \gamma(1-\psi\tau)Ak^{\alpha}; \tag{16}$$

$$y = Ak^{\alpha}; \tag{17}$$

where π, k, z and y denote steady state values of the respective variables.

Equation (15) can be written as

$$k = \mathtt{G}(k);$$

where

$$\mathtt{G}(k) = \frac{\pi(k)}{1+\pi(k)} \Gamma k^{\alpha};$$

and $\Gamma = A \cdot (1-\tau)(1-\alpha)$ is a constant.

Under (9), at $k = 0$,

$$\mathtt{G}(0) = \frac{\underline{\pi}}{1+\underline{\pi}} \Gamma (0)^{\alpha} = 0;$$

implying that a trivial steady state exists at $k = 0$.

If π, the survival probability was constant, then $\mathtt{G}(k)$ would represent a standard concave neoclassical growth mapping, with $\mathtt{G}'(0) = \infty$, $\mathtt{G}''(k) < 0 \ \forall k$, so that a unique interior steady state exists and the dynamics would be globally stable.

However, with endogenous survival probability, other possibilities exist.

Lemma 1 $\lim_{k\to 0} \pi'(k) < \infty$ *is a sufficient condition for* $\mathtt{G}'(0) = 0$.

Proof: Note that π is continuous and differentiable in its arguments which in turn are continuous and differentiable in k. Therefore, π is continuous and differentiable in k and

$\mathtt{G}(k)$ is continuous and differentiable in k. Taking derivatives of both terms in $\mathtt{G}(k)$ and rearranging:

$$\mathtt{G}'(k) = \left[\frac{\Gamma k^{\alpha}}{1+\pi(k)}\right]\left[\alpha\frac{\pi(k)}{k} + \frac{\pi'(k)}{1+\pi(k)}\right]; \tag{18}$$

it can be seen that the shape of $\mathtt{G}(k)$ can be quite different from the standard neoclassical mapping, depending on how $\pi'(k)$ varies with k. Taking the limits of the two terms inside square brackets as $k \to 0$, the first term clearly goes to zero and the limit of the second term inside square brackets can be expressed as

$$\alpha \cdot \left\{\lim_{k\to 0}\frac{\pi(k)}{k}\right\} + \left\{\lim_{k\to 0}\frac{\pi'(k)}{1+\pi(k)}\right\};$$

where the limit of the first term inside curly brackets is given by L'Hopital's Rule as:

$$\lim_{k\to 0}\frac{\pi(k)}{k} = \lim_{k\to 0}\pi'(k).$$

It can be seen that $\lim_{k\to 0}\pi'(k) < \infty$ is a sufficient condition for both the terms inside curly brackets to remain finite so that limit of $G'(k)$ approaches zero as $k \to 0$. ■

Lemma 1 rules out an Inada condition in the reduced-form relationship between survival probability and its determinants. In its absence, it is possible that $\mathtt{G}'(0) > 1$ and a unique steady state with globally stable dynamics would result, as in a standard neoclassical growth model. While Lemma 1 implies that for low values of k: $k > \mathtt{G}(k)$, the reverse is true for sufficiently large values of k. If we let $\tilde{k} = (0.5\Gamma)^{\frac{1}{1-\alpha}}$ for given Γ, α; then $\forall k \geq \tilde{k}$, $\mathtt{G}(k) \leq k$. To see this, suppose $k \geq \tilde{k}$ and that, contrary to the claim, $\mathtt{G}(k) > k$. Since $\pi \leq 1$ by definition, then $\pi/(1+\pi) \leq 0.5$ and $\mathtt{G}(k) \leq 0.5\Gamma k^{\alpha}$. By transitivity it must be the case that $0.5\Gamma k^{\alpha} > k$. But then $0.5\Gamma > k^{1-\alpha}$ and $(0.5\Gamma)^{\frac{1}{1-\alpha}} \equiv \tilde{k} > k$, leading to a contradiction.

So far we have established that either (i) there is no interior steady state or (ii) there are multiple interior steady states. To ensure (ii), note that the steady state equation can be rearranged as follows:

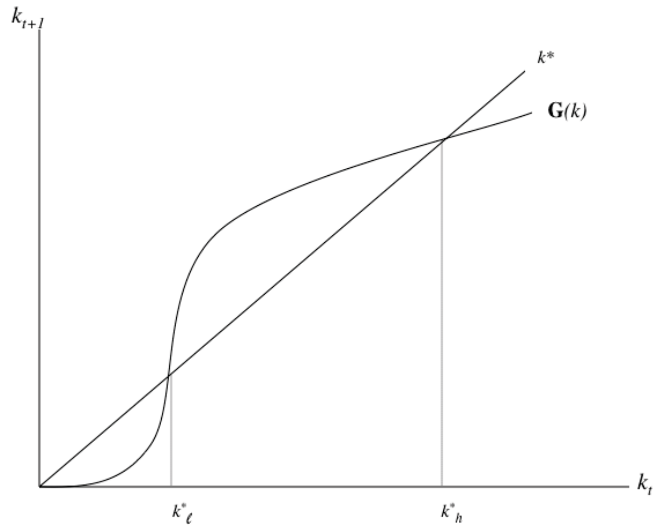
$$\Gamma = \frac{1+\pi(k)}{\pi(k)}k^{1-\alpha}.$$

Given the function $\pi(k)$ and any finite and positive value of k, the right-hand side will be positive and finite. Since Γ is exogenous and positively related to A for $\tau < 1$ and $\alpha < 1$, there always exists A large enough that

$$\Gamma > \frac{1+\pi(k)}{\pi(k)}k^{1-\alpha}.$$

This leads to the following result, stated without proof:

Figure 1: Multiple steady states



Lemma 2 *For any $\alpha \in (0,1)$ and $\tau \in (0,1)$ there exists an $\hat{A} < \infty$ and a $\hat{k} < \infty$ and asssociated $\hat{\Gamma}$: $\hat{\Gamma} = ((1+\pi(\hat{k}))/(\pi(\hat{k}))\hat{k}^{1-\alpha}$, such that $\Gamma > \hat{\Gamma}$, $\mathtt{G}(\Gamma, \hat{k}) > \hat{k}$.*

Lemma 1 implies that so long if total factor productivity (TFP) is high enough (given a function $\pi(k)$), $\mathtt{G}(k)$ will exceed k for a non-empty interval of values of k. Along with the results on the slope and level of $\mathtt{G}(k)$ derived earlier, this leads to the following proposition

Proposition 1 *If TFP, A is large enough, and Assumption 2 holds, then there are two interior steady states, k^*_ℓ and k^*_h, such that $k^*_\ell < \hat{k} < k^*_h$.*

The higher steady state, k^*_h has more capital and therefore more consumption as well as a higher stock of pollution. Despite this, it offers a greater survival probability. In the steady state, the survival probability is

$$\pi(k) = \frac{k^{1-\alpha}}{\Gamma - k^{1-\alpha}};$$

which is increasing in k.

Figure 1 below represents the transition map, depicting k_{t+1} as an S-shaped function of k_t for a given tax rate, τ.

The 45^o line represents potential steady states. $\mathtt{G}(k)$ is S-shaped upwards, sharing its origin with the 45^o line and intersecting it at two other points k^*_ℓ, k^*_h. Since, for points which lie between the origin and k^*_ℓ, $\mathtt{G}(k)$ lies below the 45^o line, any path starting off with $k_0 \in (0, k^*_\ell)$ will converge to the trivial steady state, while for points between k^*_ℓ and k^*_h, $\mathtt{G}(k)$ lies above the 45^o line, any path starting off at $k_0 > k^*_\ell$ will converge to k^*_h.

k^*_ℓ represents a poverty trap not just in the sense that it is the steady state with lower levels of economic activity and pollution flows, but also in the sense that it represents a threshold below which the equilibrium path of the economy converges asymptotically towards zero. We shall therefore refer to this type of steady state as a 'poverty trap'. k^*_h represents a stable steady state, which resembles locally the unique steady state of a neoclassical growth model. We shall refer to this type of steady state as a 'neoclassical steady state' even when it is paired with a poverty trap.

The concavity of $\mathtt{G}(k)$ can lead to it sloping downward at some point. A necessary condition is $\pi'(k) < 0$, which can happen at high enough values of k. This can lead to oscillations and limit cycles in the stock of capital and the flow of pollution around the upper steady state.[16] In the subsequent sections we assume that this condition does not hold to highlight the role of optimal taxes on dynamics.

3.2 Varying τ

To understand the dynamic effect of abatement policy on growth, we differentiate the steady state transition mapping, $\mathtt{G}(k)$, with respect to τ:

$$\left.\frac{\partial \mathtt{G}(k)}{\partial \tau}\right|_k = \left[-\frac{\pi}{1+\pi} - \frac{\pi_z \gamma \psi (1-\tau) A k^\alpha}{(1+\pi)^2}\right](1-\alpha)Ak^\alpha; \tag{19}$$

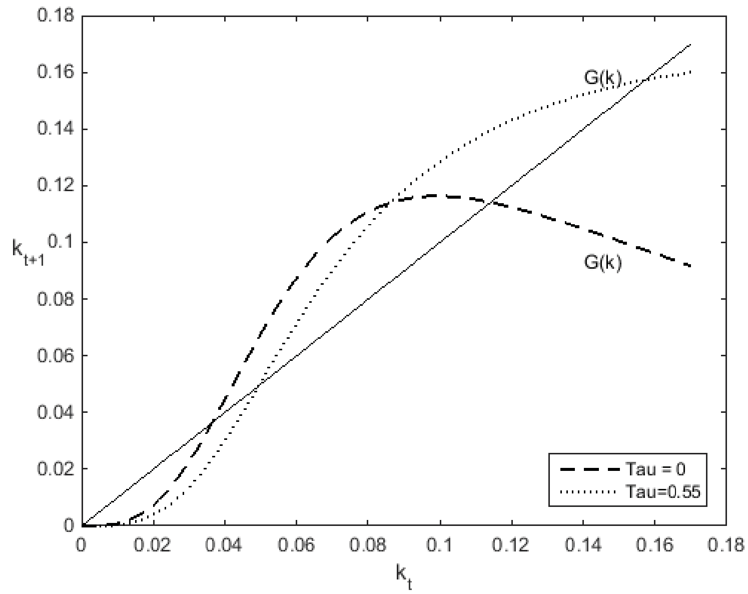
where π_z is the partial of π with respect to z alone (the effect of k on z is accounted for by the rest of the numerator in the second term). The above derivative is ambiguous in sign because $\pi_z < 0$. An increase in τ lowers net wage incomes, which at constant π shifts $\mathtt{G}(k)$ downwards. On the other hand, a higher τ raises π via the abatement effect on z. This tends to work against the downward shift in $\mathtt{G}(k)$. But the latter effect is weighted by k^α and is likely to be dominated by the direct effect of τ on wage income at low values of k. Thus $\mathtt{G}(k)$ is likely to shift down at low levels of k but it might shift up at higher levels. We next give an example of survival probability function to generate these comparative static effects.

In Appendix A2, we consider variants of the following functional form

$$\pi = \frac{\underline{\pi} + y^\vartheta}{1+y^\vartheta}\frac{1}{1+z^\delta};$$

[16] Note that $\mathtt{G}(k)$ cannot slope downwards at the low steady state, even if $\pi'(k) < 0$.

Figure 2: A uniform increase in the tax rate.



and discuss conditions under which they would lead to Assumption 2 being satisfied and for multiple steady states to arise. We then use the following parameter values

$$\alpha = 1/3, A = 2.4, \gamma = 1.11, \underline{\pi} = 0.0, \vartheta = 9, \delta = 5, \psi = 0.8;$$

and using MATLAB trace out the transition map of the capital stock for $\tau = 0$ and $\tau = 0.55$. The results are depicted in Figure 2.

The original steady states, at $\tau = 0$, are $k_\ell = 0.035$ and $k_h = 0.114$ respectively. An increase in the abatement tax to $\tau = 0.55$ causes a downward shift in $\mathtt{G}(k)$ at low levels of capital stock but upwards at the high capital stock. There are two new steady states, $k_\ell^{*\prime} = 0.050$ and $k_h^{*\prime} = 0.158$ respectively. Compared with their respective predecessors, *both* steady states have higher levels of capital stock. The dynamic implication is that the basin of attraction of a trivial steady state has now increased, while economies that start off to the right of $k_\ell^{*\prime}$ can now converge to a higher steady state than before. Thus, with an increase in the exogenous tax, it is possible that long-run cross-country inequality will increase.

4 Optimal taxes

We examine the social planner's problem of choosing a sequence of optimal abatement taxes to maximise the weighted sum of lifetime utilities of each generation born at time

$t+i$, $i \geq 0$, with $0 \leq \beta < 1$ representing the inter-generational discount factor. The welfare function is

$$W_t = \pi_{t-1} c_t^o + \sum_{i=0}^{\infty} \beta^i U_{t+i};$$

where

$$U_{t+i} = \ln c_{t+i}^y + \pi_t \ln c_{t+i+1}^o; \;\; i \geq 0.$$

The planner imposes a sequence of wage taxes $\{\tau_{t+i} w_{t+i}\}_{i=0}^{\infty}$ to maximise the above.

A wage tax is the natural policy instrument in the model. Ours is a one-sector model with only one choice variable for private agents, namely savings for old age. The only possible instruments are taxes on output, capital and wages.[17] An output tax, because of constant returns to scale, amounts to a uniform tax on wage and capital incomes. Taxing capital incomes is problematic as it makes the old in the initial period worse off. Hence, only wage taxes have the potential to be weakly welfare-increasing, albeit in a second-best way because of their effects on savings. Likewise, the planner is constrained to non-negative tax rates as any subsidy to the current young can only come at the expense of the current old.

Since the planner's policies are, by construction, welfare-neutral with respect to the surviving old at time t, we confine our attention to a truncated welfare function $\tilde{W}_t$ that excludes time t old. It is well known that in the absence of viable commitment strategies, the path of optimal taxes in an overlapping-generations economy may be time-inconsistent (Erosa and Gervais [2001]). To avoid this, we use dynamic programming to formulate each period's policy choice as a function of the state of the economy.

$$\tilde{W}_t = V(k_t) = \max_{\tau_t} \left[U_t + \beta V(k_{t+1}) \right].$$

Plugging in private decisions regarding c_t^y, c_{t+1}^o and k_{t+1} from equations (11), (12) and (13) respectively into the objective function, we have

$$V(k_t) = \max_{\tau_t} \left[\ln \left(\frac{(1-\tau_t)(1-\alpha)A k_t^{\alpha}}{1+\pi(k_t)} \right) + \pi(k_t) \ln \left(\frac{\hat{A}(1-\tau_t)^{\alpha} k_t^{2\alpha}}{\pi(k_t)^{1-\alpha}(1+\pi(k_t))^{\alpha}} \right) + \beta V(k_{t+1}) \right];$$

[17] We do not have a 'dirty' sector which could be taxed to fund transfers to a 'green' sector; neither do our agents have access to technologies that might offset pollution. So the type of Pigouvian taxes that can tilt incentives towards green activities are not available in our model. Some of the related papers in the literature, *e.g.* John and Pecchenino [1994], Mariani *et. al.* [2010] consider private abatement activity. This is not applicable in our model since the pollution externality arising from agents' savings decisions is passed on to agents not alive at the time the decisions are made. It should also be noted that in their papers, it is always welfare-improving to tax polluting activities and encourage abatement but in our model, reducing pollution may not improve welfare, given the dual external determinants of mortality.

where $\hat{A} \equiv \alpha(1-\alpha)^{\alpha} A^{1+\alpha}$ is a constant

Taking the first-order condition:

$$\frac{\partial V_t}{\partial \tau_t} = \Omega_t \frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} + \beta \frac{\partial V_{t+1}}{\partial k_{t+1}} \frac{\partial k_{t+1}}{\partial \tau_t} \leq 0; \tag{20}$$

where

$$\Omega_t = \ln c^o_{t+1} - \frac{2-\alpha+\pi_t}{1+\pi_t};$$

and <0 implies a zero tax.

Next, taking the derivative $\partial V_t / \partial k_t$ of the value function at time t and updating it by one period, we get

$$\frac{\partial V_{t+1}}{\partial k_{t+1}} = \frac{\alpha(1+2\pi_{t+1})}{k_{t+1}} + \Omega_{t+1} \frac{\partial \pi_{t+1}}{\partial k_{t+1}}.$$

Finally taking into account the dependence of k_{t+1} on τ_t via equation (13),

$$\frac{\partial k_{t+1}}{\partial \tau_t} = \frac{A(1-\alpha)k_t^{\alpha}}{1+\pi_t} \left[\left(\frac{1-\tau_t}{1+\pi_t} \right) \frac{\partial \pi_t}{\partial \tau_t} - \pi_t \right].$$

Putting everything together we can express the first-order condition as

$$\frac{\partial V_t}{\partial \tau_t} = \Omega_t \frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} + \beta \left[\frac{\alpha(1+2\pi_{t+1})}{k_{t+1}} + \Omega_{t+1} \frac{\partial \pi_{t+1}}{\partial k_{t+1}} \right] \left[\frac{A(1-\alpha)k_t^{\alpha}}{1+\pi_t} \left\{ \left(\frac{1-\tau_t}{1+\pi_t} \right) \frac{\partial \pi_t}{\partial \tau_t} - \pi_t \right\} \right]. \tag{21}$$

The terms in equation (21) represent the following effects: (i) the direct effects of a tax on the wage income of the current young, (ii) the indirect effects working through induced changes in survival probability and (iii) the intergenerational spillover induced by the effect of current taxes on the capital stock available to the next generation's young workers. The direct effects reduce both consumption and savings by the young, and are negative. These are captured by the second term in the optimality condition.

The indirect effects are captured in the term inside square brackets. An environmental tax raises survival probability, leading to higher expected utility in old age. At the same time the higher survival probability reduces actual consumption at both young and old age, the first because savings are increasing in survival probability; the second because although individuals save more the return to their annuities yields less because of the higher survival ratio of the population. This effect can be confirmed from equation (21) in which the term capturing the optimal old-age consumption is decreasing in π. The intuition is that while per-capita old-age capital increases by a factor of $[\pi/(1+\pi)]^{\alpha}$, the market return on a unit annuity decreases by a factor $1/\pi$.

Finally, the intergenerational effect depends on a combination of three factors: the effect of a current abatement tax on capital stock in the next period; the effect of a higher capital stock next period on the lifetime utility of the next generation and the magnitude of the intergenerational discount factor. The first two of these effects are both ambiguous, consisting themselves of further sub-effects, but whatever their sign, their magnitude is proportional to the intergenerational discount factor β.

Before proceeding to further disentangle these effects we shall first consider the case of $\beta = 0$: this is the case of a myopic government concerned only with the welfare of a single contemporaneous generation. This is a benchmark case which yields tractable results that are extended to the general case.

4.1 The myopic social planner:

When $\beta = 0$, first-order condition (20) reduces to

$$\frac{dV_t}{d\tau_t} \equiv \mathtt{H} = \left[\ln c^o_{t+1} - \frac{2-\alpha+\pi_t}{1+\pi_t} \right] \cdot \frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t} \leq 0; \tag{22}$$

where < 0 implies $\tau_t = 0$.

With some further manipulations to be described below, the above condition will underly a policy function, $\tau_t = h(k_t)$. Substituting the solution into equation (13) for capital accumulation yields $k_{t+1} = \mathtt{G}(k_t, h(k_t))$. The dynamic path of the economy is traced out by repeated iteration of the above. A steady state of the economy with optimal taxes is given by a pair k and $\tau = h(k)$ such that $k = \mathtt{G}(k, h(k))$.

Proposition 2 *If k_0 is below some threshold level $\underline{k}$, then the optimal environmental tax, $\tau^* = 0$.*

Proof: From (22) we see that a necessary condition for $\tau^* > 0$ is

$$\Omega_t = \left[\ln c^o_{t+1} - \frac{2-\alpha+\pi_t}{1+\pi_t} \right] > 0.$$

At low levels of initial capital, k_0, this is not going to hold. This is because the negative term in Ω_t is always non-zero while the positive term approaches minus infinity, given the logarithmic specification, as the capital stock approaches zero. Thus there exists some threshold level $\underline{k}$; such that for any $k_0 < \underline{k}$, $\Omega < 0$. ■

To see the potential for a positive tax at higher levels of capital, consider how Ω behaves as capital rises, abstracting for now from the equilibrium path. In principle, there will always be an arbitrarily high level of k_t such that $\Omega_t > 0$. This is because the first term in Ω_t has the potential to increase monotonically with k_t, at least after some threshold, while the second term is always bounded in the interval $[(3-\alpha)/2, (2+\underline{\pi}-\alpha)/(1+\underline{\pi})]$ and within this interval, it falls with increases in π_t. c^o_{t+1} rises monotonically with k_t even when π_t rises as well. If along the dynamic path, the detrimental effects of pollution make π_t start declining in k_t, then c^o_{t+1} rises even faster with k_t. At some level of development, Ω_t will be positive and increasing in capital. The other negative term in the first-order condition is similarly bounded above at $(1+\alpha)$, when evaluated at a zero tax rate. Thus, at a second critical level of development, an interior solution will arise for a positive optimal tax. The question is what level of development has to be reached before it arises and to what extent this level coincides with potential steady states of the economy.

To pursue these conjectures more rigorously, we first establish some general conditions for the applicability of a positive environmental tax at some threshold level of income. Let the right-hand side of equation (22) be denoted by:

$$\mathtt{H}(k_t, \tau_t) = \Omega_t \cdot \frac{\partial \pi_t}{\partial \tau_t} - \frac{1+\alpha\pi_t}{1-\tau_t}.$$

The first condition needed for a well-behaved tax function is

$$\left.\frac{\partial \mathtt{H}}{\partial \tau_t}\right|_{\mathtt{H}=0} < 0.$$

In other words, that the second-order condition is satisfied whenever the first-order condition holds as an equality.

The second condition ensuring a well behaved tax function is:

$$\left.\frac{\partial \mathtt{H}}{\partial k_t}\right|_{\tau=0,\mathtt{H}=0} > 0.$$

Thus, evaluated at the point where the first-order condition first holds with equality at a zero tax, it is upward sloping in k_t. Note that at very low levels of the capital stock this may not be true, but what is required is that it holds in the neighbourhood of the threshold where an optimal tax first arises.

To explore the above conditions further, differentiate $\mathtt{H}$ with respect to its arguments (time scripts will be suppressed as all variables are contemporaneous. After some manipulation, these derivatives can be written as

$$\frac{\partial \mathtt{H}}{\partial \tau} = \Omega\frac{\partial^2 \pi}{\partial \tau^2} - \frac{2\alpha}{1-\tau}\frac{\partial \pi}{\partial \tau} - \frac{1+\alpha\pi}{(1-\tau)^2} - \frac{\pi(1+\pi)+(1-\alpha)}{\pi(1+\pi)^2}\left(\frac{\partial \pi}{\partial \tau}\right)^2; \quad (23)$$

$$\frac{\partial \mathtt{H}}{\partial k} = \frac{\partial \Omega}{\partial k}\frac{\partial \pi}{\partial \tau} + \Omega\frac{\partial^2 \pi}{\partial \tau \partial k} - \frac{\alpha}{1-\tau}\frac{\partial \pi}{\partial k}; \quad (24)$$

where

$$\frac{\partial \Omega}{\partial k} = \frac{2\alpha}{k} - \frac{(1+\pi)^2-\pi-\alpha}{(1+\pi)^2}\nu_{\pi k};$$

where $\nu_{\pi k}$ is the elasticity of survival probability with respect to capital. This is eventually decreasing in k due to the positive and eventually diminishing effects of greater income and the negative and eventually increasing effects of higher pollution. It can turn negative at some point; however, we shall restrict our analysis to cases where it remains strictly positive.

None of the above terms can be signed unambiguously but two comments are in order. First, as noted before, a positive effect of k on Ω is necessary for the first-order condition to eventually hold. What this in turn requires is that along the infra-marginal path of capital, *i.e.* before the first-order condition kicks in, there is some range of values of k where the elasticity of survival probability with respect to the capital stock (taking into account both the beneficial and detrimental effects) is sufficiently small. As noted above, this elasticity will eventually diminish with growth in the capital stock, implying the existence of a threshold value of capital after which $\partial\Omega/\partial k > 0$.

Proposition 3 *There exists $\tilde{k} > 0$, such that for all $k > \tilde{k}$, $\dfrac{\partial\Omega}{\partial k} > 0$.*

From hereon we neglect consideration of values of k below this threshold, as for the purposes of deriving an environmental tax, such values of k cannot admit positive solutions of τ. Second, a sufficient condition for the second-order condition for τ to be negative is that π is concave in τ. However, this is likely to be too restrictive, given the following relationship between the second-order derivatives of π with respect to τ and z:

$$\frac{\partial^2\pi}{\partial\tau^2} = (\psi\gamma A k^\alpha)^2 \frac{\partial^2\pi}{\partial z^2}.$$

Thus, π will be concave in τ if and only if it is downwards concave in z. But given the likely impact of pollution levels on survival probability, this portion of the $\pi - z$ relationship applies at lower levels of pollution, when it is less likely that the first-order condition for an optimal tax will hold as an equality. At higher levels, it is unlikely that π is concave in τ. This rules out imposing concavity on the $\pi - \tau$ relationship as a sufficient condition for ensuring the validity of the second-order condition.

To proceed further, we turn to the specific example of the survival probability assumed earlier.

$$\pi = \pi^A \pi^B = \left[\frac{\underline{\pi} + y^\vartheta}{1 + y^\vartheta}\right] \left[\frac{1}{1 + z^\delta}\right]$$

In the following subsections we first analyse the sign of $\partial^2\pi/\partial\tau^2$ and then the sign of $\partial^2\pi/(\partial\tau\partial k)$

4.1.1 The second-order condition, $\partial \mathtt{H}/\partial\tau$

The following expressions are derived for the specific functional form for π (time scripts are again suppressed).

$$\begin{aligned} \frac{\partial\pi}{\partial\tau} &= \pi^A \frac{\psi\delta\gamma A k^\alpha z^{\delta-1}}{(1+z^\delta)^2} > 0; \\ \frac{\partial^2\pi}{\partial\tau^2} &= \pi^A \frac{(\psi\gamma A k^\alpha)^2 \delta z^{\delta-2}}{(1+z^\delta)^3} \left[(\delta+1)z^\delta - (\delta-1)\right]. \end{aligned} \tag{25}$$

By comparing the two expressions, the latter can be written as

$$\frac{\partial^2\pi}{\partial\tau^2} = \pi^A \left(\frac{\psi\gamma A k^\alpha \delta}{z(1+z^\delta)} \cdot \frac{\partial\pi}{\partial\tau}\right) \left[(\delta+1)z^\delta - (\delta-1)\right] \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} 0 \text{ as } z^\delta \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} \frac{\delta-1}{\delta+1}; \tag{26}$$

confirming the dependence of the sign of $\partial^2\pi/\partial\tau^2$ on that of $\partial^2\pi/\partial z^2$. To proceed further with an analysis of the second-order condition, i.e. equation (23), note from equation (4) that:

$$\gamma A k_t^\alpha = \frac{z_t}{1-\psi\tau_t}.$$

Suppressing time subscripts, let us write this as

$$\gamma A k^\alpha = \frac{z}{1-\psi\tau}.$$

Then (26) can be further modified:

$$\frac{\partial^2\pi}{\partial\tau^2} = \pi^A \left(\frac{\psi\delta z}{z(1-\psi\tau)(1+z^\delta)} \cdot \frac{\partial\pi}{\partial\tau}\right) \left[(\delta+1)z^\delta - (\delta-1)\right]$$

Now, from equation (22),

$$\Omega \leq \frac{1+\alpha\pi}{1-\tau} \frac{1}{\partial\pi/\partial\tau}, \; \forall\, \tau$$

Thus, taking the term involving $\partial^2\pi/\partial\tau^2$ in equation (23),

$$\Omega \frac{\partial^2\pi}{\partial\tau^2} \leq \left(\frac{1+\alpha\pi}{1-\tau} \frac{\psi\delta z}{z(1-\psi\tau)(1+z^\delta)}\right) \left[(\delta+1)z^\delta - (\delta-1)\right]$$

Combining with one of the other terms in equation (23)

$$\Omega \frac{\partial^2\pi}{\partial\tau^2} - \frac{1+\alpha\pi}{(1-\tau)^2} \leq \left[\frac{1+\alpha\pi}{1-\tau}\right] \left[\frac{\psi\delta z - \left[(\delta+1)z^\delta - (\delta-1)\right]}{z(1-\psi\tau)(1+z^\delta)} - \frac{1}{1-\tau}\right] \tag{27}$$

The sign of the above term will depend on the sign of the term inside square brackets. After some manipulation, the sign of the latter can be shown to be negative if the following holds:

$$-\frac{[1-\psi\{1+\delta(1-\tau)\}]z^{\delta}}{(1-\psi\tau)(1+z^{\delta})(1-\tau)} < 0$$

A sufficient condition for the above term to be negative for all values of endogenous variables is $\psi < 1/(1+\delta)$.[18]

We have therefore established:

Lemma 3 *A sufficient condition for $\partial\mathtt{H}/\partial\tau$ to be negative at all values of endogenous variables and along the entire dynamic path is $\psi/(1+\delta) < 1$.*

Recall that $\psi = \dfrac{\chi(1-\alpha)}{\gamma}$, where χ is the effectiveness of the abatement technology γ is how polluting is the productive activity. As we would expect, if the first is low enough and/or the second high enough, then the second order condition holds, or in other words there is an interior solution.

4.1.2 The sign of $\partial\mathtt{H}/\partial k$

Note the following derivatives for the assumed functional form (time indices continue to be suppressed):

$$\begin{aligned}
\frac{\partial\pi^{A}}{\partial k} &= \frac{\alpha}{k}\frac{\vartheta(1-\underline{\pi})y^{\vartheta}}{(1+y^{\vartheta})^{2}}; \\
\frac{\partial\pi^{B}}{\partial k} &= -\frac{\alpha\gamma(1-\psi\tau)Ak^{\alpha}}{k}\frac{\delta z^{\delta-1}}{(1+z^{\delta})^{2}}.
\end{aligned}$$

Using the definitions of π^{A}, π^{B}, and π, and rearranging, we can combine the above derivatives

$$\frac{\partial\pi}{\partial k} = \frac{\alpha\pi}{k}\left[\frac{\vartheta(1-\underline{\pi})y^{\vartheta}}{(1+y^{\vartheta})(\underline{\pi}+y^{\vartheta})} - \frac{\delta z^{\delta}}{(1+z^{\delta})}\right]; \tag{28}$$

which implies that

$$\nu_{\pi k} = \alpha\left[\frac{\vartheta(1-\underline{\pi})y^{\vartheta}}{(1+y^{\vartheta})(\underline{\pi}+y^{\vartheta})} - \frac{\delta z^{\delta}}{(1+z^{\delta})}\right]$$

[18] By extending the comparison with the sign of $\Omega\cdot\partial^{2}\pi/\partial\tau^{2}$ to other terms in the expression for $\partial^{2}\mathtt{H}/\partial\tau^{2}$ even weaker conditions can be derived. But as with the above, to ensure negativity of the second-order condition for all admissible values of endogenous variables, the above condition still applies.

where $\nu_{\pi k}$ has been defined as the *elasticity* of π with respect to k.[19]

Now, to derive the sign of $\partial^2 \mathtt{H}/(\partial\tau\partial k)$, we proceed in two steps. We first derive an expression for $\partial^2\pi/(\partial\tau\partial k)$ and then use it to evaluate the sign of $\partial^2 \mathtt{H}/(\partial\tau\partial k)$.

The first step is accomplished by taking the total derivative of $\partial\pi/\partial\tau$, equation (25), with respect to k. After imposing some definitions and equalities, and rearranging terms, it can be shown that:

$$\frac{k}{\partial\pi/\partial\tau}\frac{\partial^2\pi}{\partial\tau\partial k} = \nu_{\pi k} + \alpha\delta\frac{z}{z(1+z^\delta)} > 0.$$

The full derivation is outlined in Appendix A1. From here it is easy to establish the following:

Lemma 4 *$\mathtt{H}(k,\tau) = 0 \Longrightarrow \partial \mathtt{H}/\partial k \geq 0$.*

Proof: First, the expression for $\partial^2\pi/\partial\tau\partial k$ implies that

$$\frac{\partial^2\pi}{\partial\tau\partial k} \geq \frac{\partial\pi}{\partial\tau}\frac{1}{k}\nu_{\pi k}.$$

Second, $\mathtt{H} = 0$ implies that

$$\Omega = \frac{1+\alpha\pi}{1-\tau}\frac{1}{\partial\pi/\partial\tau}.$$

Therefore, referring to equation (24),

$$\Omega\frac{\partial^2\pi}{\partial\tau\partial k} = \frac{1+\alpha\pi}{1-\tau}\frac{1}{\partial\pi/\partial\tau}\frac{\partial^2\pi}{\partial\tau\partial k} \geq \frac{1+\alpha\pi}{1-\tau}\frac{1}{k}\nu_{\pi k}.$$

Now, referring to the negative term in equation (24),

$$\frac{\alpha}{(1-\tau)}\frac{\partial\pi}{\partial k} = \frac{\alpha\pi}{(1-\tau)k}\nu_{\tau k}.$$

Combining the two terms in equation (24),

$$\Omega\frac{\partial^2\pi}{\partial\tau\partial k} - \frac{\alpha\pi}{(1-\tau)k}\nu_{\pi k} \geq \frac{1+\alpha\pi}{1-\tau}\frac{1}{k}\nu_{\pi k} - \frac{\alpha\pi}{(1-\tau)k}\nu_{\pi k} \geq \frac{1}{(1-\tau)k}\nu_{\pi k} \geq 0$$

.

■

Note that we have derived the above result for all values of τ. Thus, as an economy's capital stock increases, the slack in $\mathtt{H}$ diminishes until finally an interior solution is reached.

[19]Throughout the analysis, we assume that $\nu_{\pi k}$ remains positive, although as we have noted before, a negative value is entirely possible under some conditions, and if it happens there can be oscillations around the high steady state.

4.1.3 Positive taxes:

We can now establish:

Proposition 4 *If* $\dfrac{\psi}{1+\delta} < 1$ *and* $k \geq \tilde{k}$ *then, there (i) exists an optimal policy function,* $\tau = h(k)$, $h : [\tilde{k}, \infty) \longrightarrow [0,1]$*; (ii)* $h(k)$ *is (weakly) increasing in* k*.*

Proof: The first part follows from the strict monotonicity of $\mathtt{H}$ in both τ and k. Since $\mathtt{H}$ is strictly decreasing in τ for all k under the assumed conditions, then for any k in the relevant interval, either (i) $\mathtt{H}(0,k) \leq 0$, or (ii) $\mathtt{H}(1,k) > 0$ or (iii) $\mathtt{H}(\tau,k) = 0$ for some $\tau \in [0,1]$. Moreover, τ uniquely solves the relevant case for $\mathtt{H}$ at given k, because for any $\tau' > \tau$, in case (i) $\tau = 0$ and $\tau' > 0$ worsens the slack in $\mathtt{H}$; in case (ii) if $\tau = 1$ then τ' lies outside the unit interval and in case (iii) since $\mathtt{H}(\tau,k) = 0$ for $\tau \in [0,1]$, then $\mathtt{H}(\tau',k) < 0$. Similar argument rules out the possibility that $\tau' < \tau$ also solves $\mathtt{H}$ for a given k.

The second part follows from

$$\left.\frac{\partial h(k)}{\partial k}\right|_{\mathtt{H}=0} = -\frac{\mathtt{H}_k}{\mathtt{H}_\tau} \geq 0;$$

while $\forall k \in [\tilde{k}, \infty)$, $\mathtt{H}(0,k) < 0 \Rightarrow h(k) = 0$ and $\mathtt{H}(1,k) > 0 \Rightarrow h(k) = 1$. ■

Note that $\mathtt{H}(k,1) < 0; \forall k$ since the negative term in $\mathtt{H}$ approaches $-\infty$ as τ approaches 1 at all values of k, so Proposition 4 implies that $h(k)$ approaches 1 only asymptotically. However, as we established in Proposition 2, at low values of k, the tax policy function has a flat portion. Proposition 4 thus implies a tax function which is non-homogenous in the capital stock, increasing at intermediate values of capital and approaching $\tau = 1$ in the limit. This is in line with conventional wisdom which suggests that the level and intensity of abatement should increase with the state of development.

4.2 Long-lived social planner:

We look at the continuation utility of future generations in the first-order condition (20) for determining the optimal tax on the current generation:

$$\beta \left[\frac{\alpha(1+2\pi_{t+1})}{k_{t+1}} + \Omega_{t+1}\frac{\partial \pi_{t+1}}{\partial k_{t+1}}\right]\left[\frac{A(1-\alpha)k_t^\alpha}{1+\pi_t}\left\{\left(\frac{1-\tau_t}{1+\pi_t}\right)\frac{\partial \pi_t}{\partial \tau_t} - \pi_t\right\}\right].$$

The term in second square brackets represents the effect of a higher current tax on next period's capital stock. A necessary condition for this to be positive is that the tax-financed increase in abatement activity increases the survival probability for the current young by enough to offset the negative income effect of the higher tax. The term in the first square brackets represents the effect of a higher stock of capital next period on the welfare of the next generation. That in turn depends in part on the effect of the higher capital stock on the survival probability of next period's young. Even if that is positive, the overall effect on their welfare might not be because of the term Ω_{t+1} which could be negative at low initial values of capital, for similar reasons as were identified in the case of Ω_t: higher survival probability raises the utility from given old-age consumption but lowers both young-age and old-age consumption levels; thus if the initial level of old-age consumption is low this contributes a negative effect. This discussion indicates that it will be difficult to assign a sign to the inter-generational effect on current optimal taxation on an *a priori* basis.

Since we have already derived using analytical methods a well-behaved tax policy function without incorporating the inter-generational effect and our main aim is to verify the intuition outlined above for how incorporating such effects might modify the policy function we proceed by way of numerical examples which map the policy function at varying levels of the steady state capital stock.

We start by defining the steady state version of the optimal tax equation

$$\left[\Omega\frac{\partial\pi}{\partial\tau} - \frac{1+\alpha\pi}{1-\tau}\right] - \beta\left[\frac{\alpha(1+2\pi)}{k} + \Omega\frac{\partial\pi}{\partial\tau}\right]\left[\frac{A(1-\alpha)k^{\alpha}}{1+\pi}\left\{\left(\frac{1-\tau}{1+\pi}\right)\frac{\partial\pi}{\partial\tau} - \pi\right\}\right] \leq 0; \qquad (29)$$

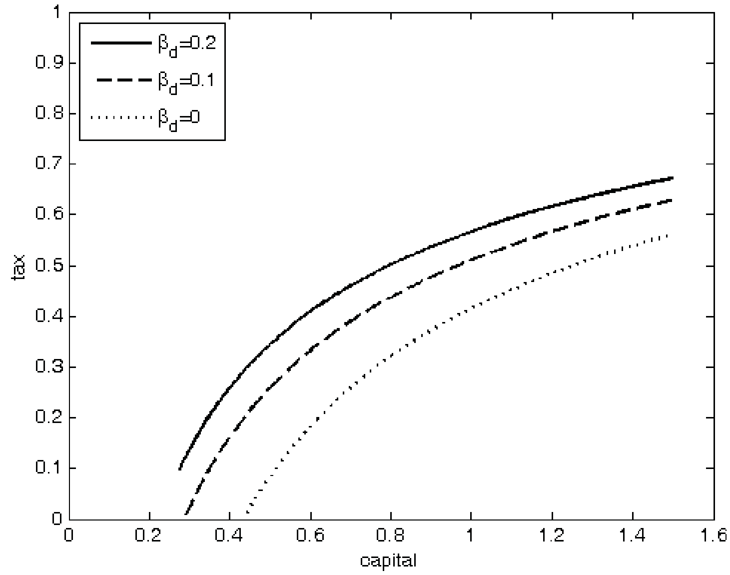
(< 0 implies that $\tau = 0$) where $\partial\pi/\partial\tau$ is given by equation (25) and $\partial\pi/\partial k$ is given by equation (28).

MATLAB was used to trace out the policy function. Taking an interval of values of steady state capital, k, equation (28) was recursively solved for the optimal value of the steady state abatement tax, τ at varying levels of the inter-generational discount factor, β. The results are in Figure 3. Other parameter values were set as in Figure 2.

We can see that the qualitative properties of the policy function are as hypothesised: regardless of the value of β, the optimal tax is zero at sufficiently low levels of k. As k rises, an upward sloping and concave tax emerges. The main effect of higher β is to shift the policy function upwards so that at any level of k the planner is more likely to undertake active abatement and to set a higher tax if positive. This is line with conventional wisdom regarding the effect of far-sighted environmental policy.

At the same time, in our model, the reaction of the capital stock to taxes can be non-convex. We have seen that at any arbitrary tax, there can be multiple steady states and that an increase in the abatement tax rate can have ambiguous effects on the steady state

Figure 3: The policy function.



capital stock. It is the interaction of a fairly conventional state-contingent environmental policy with the behaviour of the capital stock longevity that can introduce non-linearities and change the dynamics.

4.3 Dynamics of the optimal tax:

A steady state with optimal taxation is characterised by two equations.

$$k = \frac{\pi(k,\tau)}{1+\pi(k,\tau)} A \cdot (1-\tau)(1-\alpha)k\alpha; \tag{30}$$

$$\tau = h(k); \tag{31}$$

where equation (30) represents the steady state reaction function of private agents and equation (31) represents the steady state policy function of the social planner. We assume that $h(k)$ satisfies Proposition 3 for both a myopic and a long-lived social planner. A solution to the above equations is represented by a pair (k^*, τ^*).

The dynamics of the economy with optimal taxes are traced out by recursive application of the tax policy function and the transition map for the capital stock. For any capital $k_t > \tilde{k}$, $\tau_t = h(k_t)$. Then, next period's capital stock follows:

$$k_{t+1} = \frac{\pi(k_t,\tau_t)}{1+\pi(k_t,\tau_t)} A(1-\tau_t)(1-\alpha)k_t^{\alpha} = \mathtt{G}(k_t,\tau_t);$$

and so on.

This represents a first-order difference equation in k_t for any arbitrary k_0. Linearising around a steady state, the local dynamics are determined by the sign and magnitude of the expression

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}_1(k^*, \tau^*) + \mathtt{G}_2(k^*, \tau^*)h'(k^*); \tag{32}$$

where $\mathtt{G}_1(k, \tau) = \mathtt{G}'(k)$, as given by equation (18) and $\mathtt{G}_2(k, \tau)$ is given by equation (19).

It is instructive to compare equation (32) with the case of exogenous abatement, in which

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}'(k^*).$$

In this case, the dynamics of the capital stock are driven by a non-time varying $\mathtt{G}(k)$ function for a given τ. In the case of optimal abatement, the $\mathtt{G}$ function shifts (in (k_{t+1}, k_t) space) each period as the tax varies along the optimal path. This generates the possibility of additional dynamic complexity arising from a dynamic tax policy. To rule out any further complexity in the exogenous-tax case, we assume that $\mathtt{G}_1(k, \tau) > 0$ throughout this section.

Define $k^* = g(\tau)$, as the value of of k^* which solves equation (30) for any admissible τ. Then $\tau^* = h(k^*)$ solves the optimal tax at this steady state.

It is easy to show that

$$g'(\tau) = \frac{\mathtt{G}_2(k^*, \tau)}{1 - \mathtt{G}_1(k^*, \tau))}.$$

Using the above, equation (32) can be expressed as:

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}_1(k^*, \tau^*) + g'(\tau^*)(1 - \mathtt{G}_1(k^*, \tau^*))h'(k^*); \tag{33}$$

where the sign of $g'(\tau^*)$ is the same as (*resp.* the opposite of) the sign of $\mathtt{G}_2(k^*, \tau_2)$, as and when $1 - \mathtt{G}'(k^*, \tau^*) > 0$ (*resp.* < 0), as in the neoclassical steady state (*resp.* as in the poverty trap).

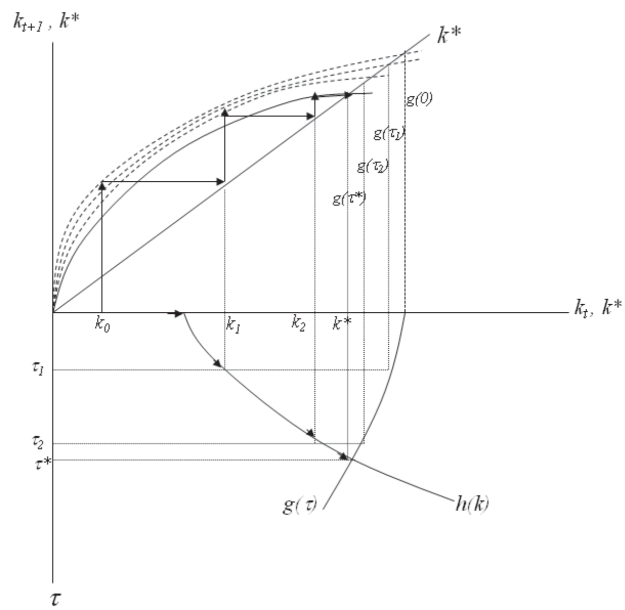
We now establish the local dynamics, first at a neoclassical steady state and then at a poverty trap.

4.3.1 Local dynamics around a neoclassical steady state

In this case, $\mathtt{G}_1(k^*, \tau^*) < 1$, $1 - \mathtt{G}_1(k^*, \tau^*) > 0$. Then $g'(\tau) > 0$ (*resp.* < 0) as $\mathtt{G}_2(k^*, \tau^*) > 0$ (*resp.* < 0). By suitable rearrangement of equation (33), it can be shown that

$$\frac{dk_{t+1}}{dk_t} \left\{ \begin{array}{c} > 1 \\ \in [0, 1] \\ < 0 \end{array} \right\} \text{as } \; g'(\tau)h'(k) \left\{ \begin{array}{l} > 1 \\ \in \left[-\frac{\mathtt{G}_1(k^*, \tau^*)}{1 - \mathtt{G}_1(k^*, \tau^*)}, 1 \right] \\ < -\frac{\mathtt{G}_1(k^*, \tau^*)}{1 - \mathtt{G}_1(k^*, \tau^*)} \end{array} \right\}.$$

Figure 4: A stable neoclassical steady state



We can see that the local dynamics around a neoclassical steady state are no longer necessarily convergent, as was the case in the exogenous tax economy. They will depend on two factors: (i) whether $g'(\tau)$ is positive or negative, *i.e.* whether an increase in the tax rate shifts the neoclassical steady state up or down; (ii) the slope of $g(\tau)$ relative to the slopes of the other two main steady state relationships: $h(k^*)$ and $\mathtt{G}_1(k^*, \tau^*)$.

Whether $g'(\tau)$ is negative or positive, but its magnitude is not too large, the dynamic path converges monotonically. When $g'(\tau)$ is positive and relatively large, the steady state becomes a source. When $g'(\tau)$ is negative and relatively large, fluctuations can arise near the steady state.

Figure 4 represents the dynamics of the stable case, with $g'(\tau)$ drawn as moderately negative. The top panel of Figure 4 shows a family of transition maps for k_{t+1} as a function of k_t. Each map is underpinned by a specific value of the optimal tax, τ_t. The lower panel depicts the functions $g(\tau)$ and $h(k)$ in $(\tau - k)$ space. $h(k)$ is always upward sloping in this space while, in keeping with the assumed nature of this steady state, $g(\tau)$ is downward sloping. Their intersection gives the combination of steady state capital and steady state taxes, (τ^*, k^*). This is the unique long-run steady state in the case depicted.

Starting at $k_0 < \underline{k}$, the latter defined in Proposition 2 as the minimum level of capital associated with active environmental policy, the optimal tax at $t = 0$ is $\tau_0 = 0$. The steady state associated with this tax is the highest dashed transition map on the top panel, which is labeled $g(0)$. If the tax rate was held constant at this level, the capital stock would evolve monotonically towards $g(0)$ through iterative application of this map. Thus at $t = 0$, next period's capital, k_1, will be given by the vertical projection to this map from k_0. But when the economy reaches k_1, the optimal tax for that period need no longer equal zero. Indeed, as drawn, the threshold level of capital is crossed and optimal $\tau_1 > 0$, as given by the projection down from k_1 to $h(k)$. At τ_1, the horizontal projection to $g(\tau)$ gives the new steady state level of capital associated with a tax rate, τ_1. This means that the transition map in the upper panel shifts downwards so it intersects the 45^o line at $g(\tau_1)$. The vertical projection from k_1 to the new transition map gives k_2 and so on. The dynamics are monotonically convergent with both k_t and τ_t rising in ever shorter steps towards the steady state.

Figure 5 shows the case of explosive dynamics in the neighbourhood of a neoclassical steady state. In this case, $g'(\tau) > 0$, *i.e.* the steady state capital stock increases with greater abatement taxes; in addition, $g'(\tau)h'(k) > 1$ so that the combined effect of an optimal tax that increases in the capital stock and the feedback from a higher tax to a higher steady state capital stock is relatively strong. Graphically, (i) both $g(\tau)$ and $h(k)$ slope upwards in $(\tau - k)$ space *and* (ii) $g(\tau)$ cuts $h(k)$ from below; hence, for any initial $k_0 > k^*$ (as shown in the diagram), $g(\tau_0) > k_0$. And since each potential steady state associated with a given (hypothetically constant) tax rate is locally stable, $k_1 > k_0$ so that the economy moves away from k^*.

Figure 6 shows the case when $g'(\tau) < 0$ and its magnitude is relatively large. As drawn, it shows the dynamic path starting at k_0 cycling between the pair (τ_0, k_0) and (τ_1, k_1) forever. This happens because $g'(\tau)$ is large in magnitude, or since we are speaking in relative terms, both (i) $h(k)$ and (ii) $\mathtt{G}_1(k^*, \tau^*)$ are quite 'flat', *i.e.* a large change in k_t induces a small increase in τ_t while a small change in k_t induces a large change in k_{t+1}. As a consequence of these features, given that the economy starts at $k_0 < k^*$, (i) $g(\tau_0) > k^*$ and (ii) $k_1 > k^* >> k_0$. But given $\tau_1 = h(k_1)$, (i) $g(\tau_1) < k^*$ and (ii) $k_2 < k^* << k_1$. Indeed, as drawn $k_2 = k_0$ so the cycle is locally stable although this is not necessarily going to be the case. The point is that oscillations can arise if these two features are present.

We summarise these results under the following Proposition, stated without further proof.

Proposition 5 *Suppose there exists a neoclassical steady state (k^*, τ^*) in an economy with optimal taxation. Then, given $\mathtt{G}_1(k^*, \tau^*)) < 1$ and $(1 - \mathtt{G}_1(k^*, \tau_1)) > 0$,*

Figure 5: A locally unstable neoclassical steady state.

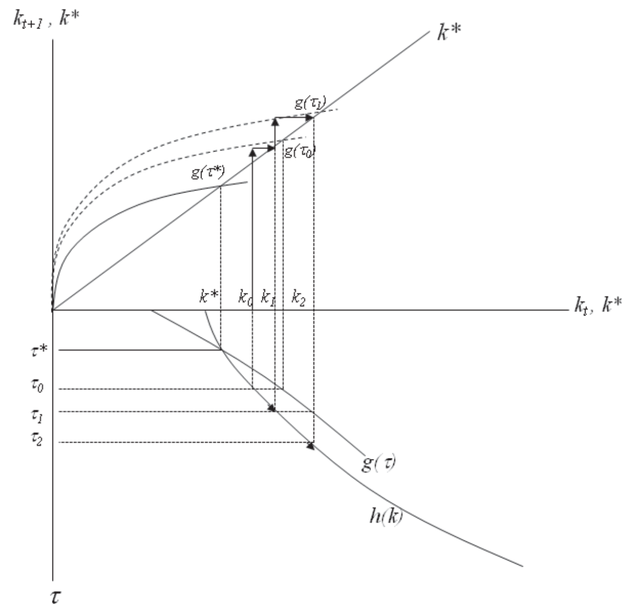
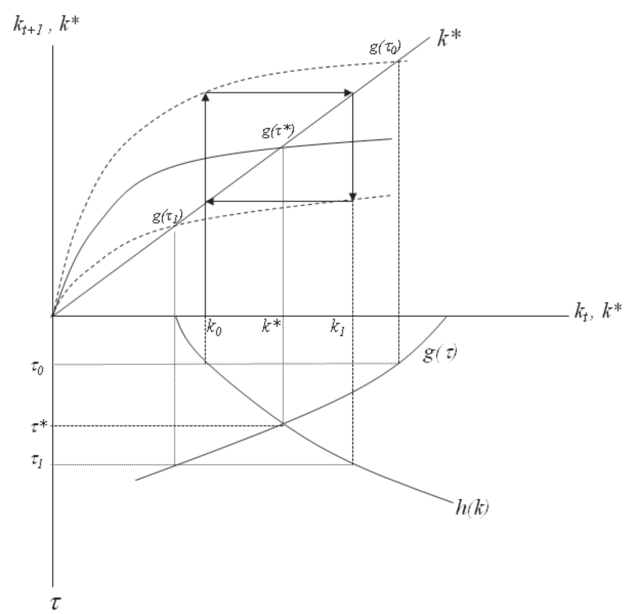


Figure 6: Oscillations around a neoclassical steady state.



(i) the steady state will be locally unstable if $g'(\tau^*)h'(k^*) > 1$*;*
(ii) there will be local fluctuations around the steady state if

$$g'(\tau^*)h'(k^*) < -\{\mathtt{G}_1(k^*,\tau^*)/(1-\mathtt{G}_1(k^*,\tau^*)\} < 0;$$

(iii) the dynamics will be monotonically convergent in all other cases.

4.3.2 Local dynamics around a poverty trap:

In this case, the steady state map cuts the 45^o line from below; therefore $\mathtt{G}_1(k^*,\tau^*) > 1$ and $1 - \mathtt{G}_1(k^*,\tau^*) < 0$. Thus $g'(\tau) > 0$ (*resp.* < 0) as $\mathtt{G}_2(k^*,\tau^*) < 0$ (*resp.* > 0). To remain consistent with the discussion following equation (19) in Section 3, we shall exclude the case $g'(\tau) < 0$ from further consideration. Thus, equation (33) can be written more clearly as

$$\frac{dk_{t+1}}{dk_t} = \mathtt{G}_1(k^*,\tau^*) - g'(\tau^*)(\mathtt{G}_1(k^*,\tau^*) - 1)h'(k^*).$$

It can now be established by suitable rearrangement that[20]

$$\frac{dk_{t+1}}{dk_t}\left\{\begin{array}{c} >1 \\ \in[0,1] \\ <0 \end{array}\right\} \text{as } g'(\tau)h'(k)\left\{\begin{array}{l} <1 \\ \in\left[1,\frac{\mathtt{G}_1(k^*,\tau^*)}{\mathtt{G}_1(k^*,\tau^*)-1}\right] \\ >\frac{\mathtt{G}_1(k^*,\tau^*)}{\mathtt{G}_1(k^*,\tau^*)-1} \end{array}\right\}.$$

Whereas a poverty trap was monotonically a source in the case of exogenous taxes, it can now be a sink. There can also be fluctuations around the poverty trap, depending on how strong the interaction of optimal policy with private sector capital accumulation decisions is.

Figure 7 shows the case of a monotonically stable poverty trap.[21] In this case, $h'(k)g'(\tau) > 1$, so that as k increases $h(k)$ cuts $g(\tau)$ from above in the lower panel.

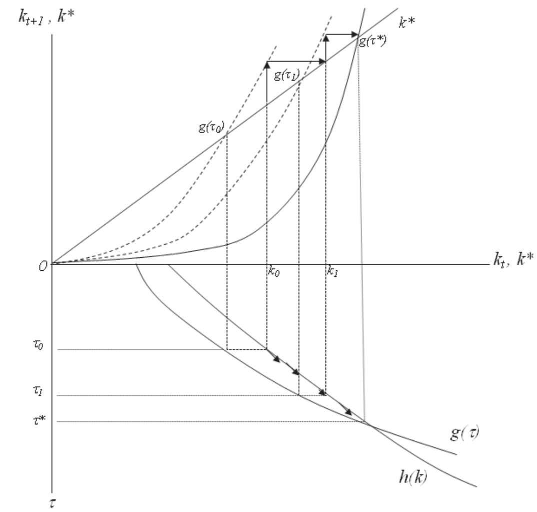
Starting at an initial capital, $k_0 < k^*$ and tax rate $\tau_0 < \tau^*$, the transition map associated with τ_0 would result in a steady state $g(\tau_0)$ which lies below k_0. Because $g(\tau_0)$ is (for constant τ) unstable, this means that $k_1 > k_0$. Then $\tau_1 > \tau_0$ and $g(\tau_1)$ lies above $g(\tau_0)$ but below k_1. Thus $k_2 > k_1$, $\tau_2 > \tau_1$ and the economy is on a path that converges to (τ^*, k^*).

Intuitively Figure 7 depicts a case in which abatement taxes become optimal only at a relatively high level of capital but are subsequently fairly sensitive to increases in capital.

[20]It is implicit in the above that for any variable $x > 1$, $x/(x-1) \to 1$ from above as $x \to \infty$.

[21]The case in which the dynamics remain qualitatively similar to the exogenous-tax case is discussed in a working paper version of this pape, [Goenka *et. al.*, 2012.]

Figure 7: A locally stable poverty trap



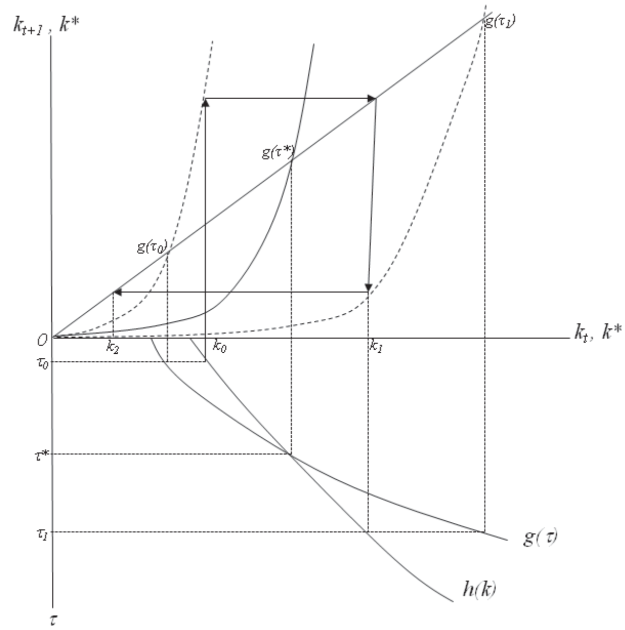
This results in $h(k)$ cutting $g(\tau)$ from above. When the initial capital stock is below the steady state, the optimal tax rate associated with that capital stock maps into an associated (transitory) steady state which lies below the initial capital stock. This results in next period's capital stock being higher than the initial one and closer to the long-run steady state.

Finally, the possibility of cycles around a poverty trap is illustrated in Figure 8. As drawn, both the policy function and the transition map are steep. Since the transition map cuts the 45^o line from below this means that k_{t+1} is quite sensitive to changes in k_t. The slope of the policy function further implies that τ_t is sensitive to changes in k_t. When the economy starts at k_0, the tax rate is τ_0 and the dynamics follows $g(\tau_0)$, along this map, capital increases by a large amount to $k_1 > k^*$. This causes the tax rate at $t = 1$ to increase to τ_1 causing a large shift in the transition map to $g(\tau_1)$. Given k_1 thenm there is a large drop in capital to $k_2 < k^*$. As drawn, the cycle is explosive; however this need not be the case; the cycle could be stable or convergent.[22]

Proposition 6 *Suppose there exists a poverty trap (k^*, τ^*) in an economy with optimal taxation. Then, given that $G_1(k^*, \tau^*) > 1$, $G_1(k^*, \tau^*)/[G_1(k^*, \tau^*) - 1] > 1$,*

[22]It is worth noting the difference with Palivos and Varvarigos [2010]; while they argue that environmental taxation can be used to eliminate cycles associated with the impact of pollution on uncertain lifetimes, our results show that second-best welfare-maximising environmental taxes can in themselves be a source of oscillations.

Figure 8: Cycles around a poverty trap.



(i) the steady state will be locally unstable if $g'(\tau^*)h'(k^*) < 1$*;*
(ii) there will be local fluctuations around the steady state if

$$g'(\tau^*)h'(k^*) > G_1(k^*,\tau^*)/[G_1(k^*,\tau^*) - 1] > 1;$$

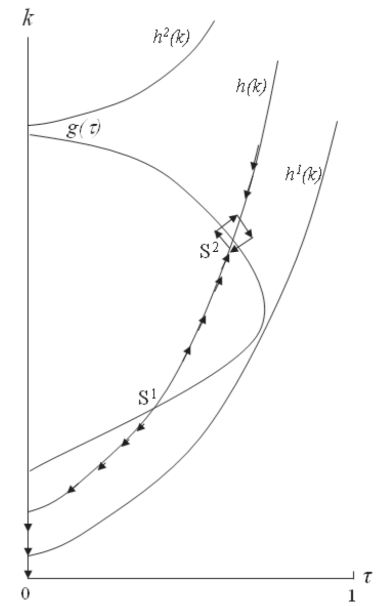
(iii) the dynamics will be monotonically convergent in all other cases.

4.3.3 Existence, uniqueness and stability of steady states:

This section makes use of the local analysis carried out above. Figure 9 depicts an economy in which multiple steady states arise at any given tax rate. Thus, the locus $g(\tau)$ is D-shaped (note that the axes have been rotated by 90^o degrees anti-clockwise in relation to Figures 4-8). The locus $h(k)$ is upward sloping and concave throughout. Because of the shape of $g(\tau)$, the existence of a steady state with optimal taxes is not guaranteed. We have drawn three different versions of the $h(k)$ locus, which for a given underlying economy, could correspond to three different values of the intergenerational discount factor.

With $h^1(k)$ there is no interior steady state associated with optimal tax policy, while with $h^2(k)$ there is zero taxation at both the low and high steady states. It is only with $h(k)$

Figure 9: Non-existence and multiplicity of steady states with optimal policy.

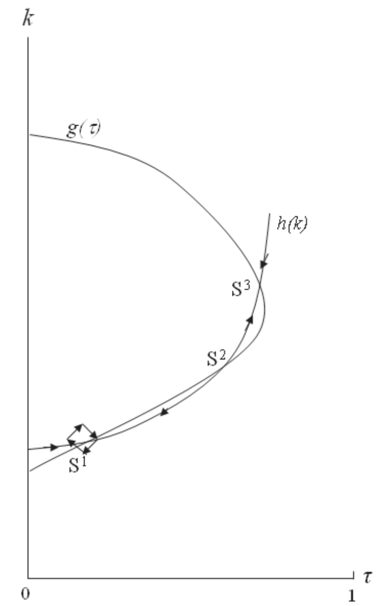


that there is positive taxation at both steady states: the lower steady state, S^1 has lower capital and lower taxation and is locally unstable. The higher steady state, S^2 is stable but as shown in Section 4.3.1, local cycles are possible around this steady state. Both possibilities are illustrated in Figure 9.

In Figure 10, we show a case where $h(k)$ cuts the k-axis at a point that lies inside the D. It then cuts $g(\tau)$ at three interior points, S^1, S^2 and S^3. Both S^1 and S^2 are poverty traps and S^3 is a neoclassical steady state. S^1 and S^3 are both stable, although both can give rise to cycles (the latter are shown only around S^1). Thus, while poverty traps are always unstable under exogenous taxation, optimal policy can render them locally stable.

Finally, Figure 11 presents another intriguing consequence of optimal policy. This figure illustrates a parametrized example computued by MATLAB using the same parameter values as the ones that generated Figure 2. In Figure 11 only the neoclassical steady state is shown. Recall that in Figure 2, we established numerically that the response of the capital stock to the tax rate in a neoclassical steady state can be upward sloping. Thus, the locus $g(\tau)$ is upward sloping reflecting that possibility. While Figure 2 was drawn for arbitrary taxes and did not depend on the intergenerational preferences of a social planner, Figure 11 illustrates the optimal tax function for $\beta = 0.9$.

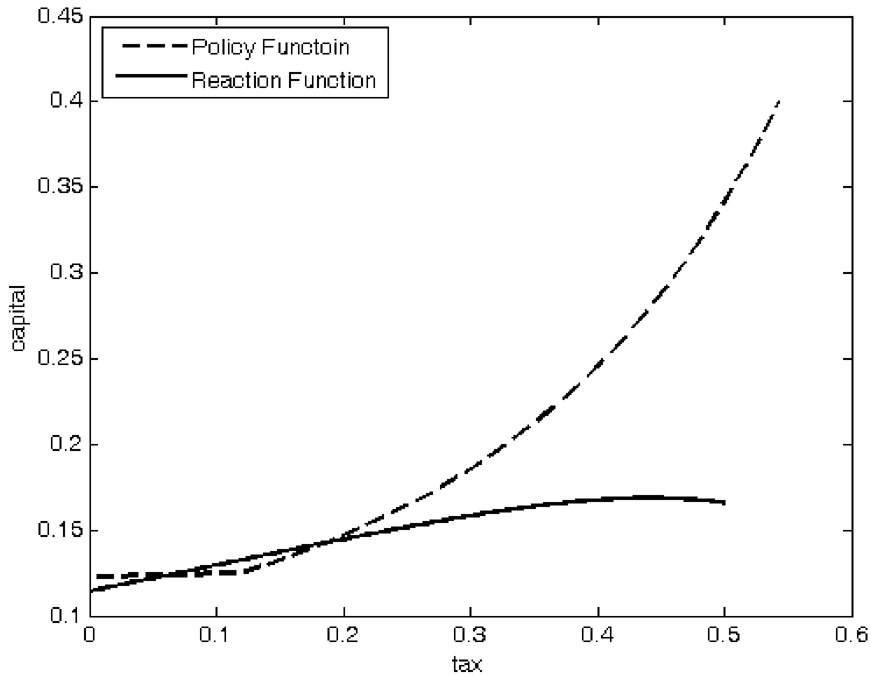
Figure 10: Multiple poverty traps.



We have established in earlier sections that an optimal second-best tax might have a positive intercept on the capital axis. Figure 11 shows that when the intercept of the tax policy function is sufficiently high, as happens in our numerical example, then there can be multiple intersections between the policy function and the reaction function, leading to two neoclassical-type steady states. Thus we have a low steady state with $\tau_1 = 0.065$, $k_{h1} = 0.124$ and a high steady state with $\tau_2 = 0.185$, $k_{h2} = 0.1422$. We label the low capital steady state as k_{h1} to avoid confusion with a poverty trap which is labelled k_ℓ. Two neoclassical steady states arise in this example as the $g(\tau)$ function which maps the tax rate into steady state levels of the capital stock is upward sloping, in line with the case depicted in Figure 2. The tax policy function $h(k)$ has the expected shape: zero for low levels of capital, and then increasing and concave.

In this figure, the low-capital neoclassical steady state is unstable while the higher-capital steady state is stable. What this suggests is that an economy that starts at a level of capital below S^1 is caught in an 'environmental trap' which results in successively lower levels of environmental controls, resulting in successively lower levels of capital.

Figure 11: A locally unstable neoclassical steady state, $\beta = 0.9$.



5 Conclusions

This paper has shown that the combined effect of income and pollution on life expectancy can lead to multiple interior steady states, with an unstable poverty trap and a stable, neoclassical steady state. We examined the comparative static effects of exogenous tax abatement policy and showed that this will widen the basin of the poverty trap sand can stimulate higher capital accumulation at the neoclassical steady state.

The main contribution of the paper has been the characterisation of the optimal environmental taxation where a forward-looking planner sets taxes taking as given the optimal saving decisions of each generation. We show that the tax is non-homogeneous and monotonically increasing in the capital stock. From a policy point of view, this suggests that economies that are close to or just emerging from a poverty trap might impose zero or low levels of environmental protection but eventually this will rise along the growth path.

More importantly, we have shown that optimal policy might itself contribute to complex dynamics in several ways: first, a steady state with optimal taxes might not exist when in the underlying economy with exogenous policy, one or more interior steady states existed; second, by inducing multiple steady states under conditions where a unique steady state would have existed with exogenous policy; third, by stabilising poverty traps which would be unstable under exogenous policy; fourth, by inducing oscillations and cycles around steady states which would otherwise be locally stable.

With respect to the last finding, we offer a word of caution. Although there is evidence that short term fluctuations in air quality can lead to fluctuations in mortality rates (see Evans and Smith [2005], Huang *et al.* [2012]), it is not clear that these phenomena are in turn part of a general business cycle or driven by seasonality. The main lesson that we would like to emphasise through these findings is that in cases such as the one we have studied, where the impact of state variables on economic outcomes is not uniformly monotonic, optimal policy itself can contribute to economic fluctuations and multiplicity of steady states, rather than reduce them. Thus, models that impose steady state conditions to derive optimal policy can be misleading about both the transitional dynamics and the asymptotic outcomes.

References

[1] Ayres, J.G. (2006) Cardiovascular disease and air pollution: A report by the medical committee on the effects of air pollution, Department of Health, UK.

[2] Azomahou, T. T., Boucekkine, R., and Diene, B. (2009) A closer look at the relationships between life expectancy and economic growth, *International Journal of Economic Theory* 5: 201-244.

[3] Bhattacharya, J. and Qiao, X. (2007) Public and Private Expenditures on Health in a Growth Model, *Journal of Economic Dynamics and Control*, 31 (8): 2519-2535.

[4] Bovenberg, A. L. and Heijdra, B.J. (1998) Environmental tax policy and intergenerational distribution, *Journal of Public Economics*, 67 (1998) 124.

[5] Chakraborty, S. (2004) Endogenous lifetime and economic growth, *Journal of Economic Theory* 116: 119-137.

[6] Cutler, D., Deaton, and A.S. Lleras-Muney, A. (2006) The determinants of mortality, *Journal of Economic Perspectives*, 20(3), 97-120.

[7] Economides, G. and Philippopoulos, A. (2008) Growth enhancing policy is the means to sustain the environment, *Review of Economic Dynamics* 11: 207-219.

[8] Erosa, A., and Gervais, M. (2001) Optimal Taxation in Infinitely-Lived Agent and Overlapping Generations Models: A Review, *Federal Reserve Bank of Richmond Economic Quarterly* Volume 87/2 Spring, 23-44.

[9] Evans, J., *et al.* (2013) Estimates of global mortality attributable to particulate air pollution using satellite imagery, *Environmental Research* 120: 33-42

[10] Evans, M.F., and Smith, V.K. (2005) Do new health conditions support mortality-air pollution effects? *Journal of Environmental Economics and Management* 50: 496-518.

[11] Georgiadis, G., Penido, J. and Rogriguez, F. (2010) Has the Preston curve broken down, Human Development Research Papers, HDRP-2010-32, United Nations Development Programme.

[12] Ghiglino, C., and Tvede, M. (2000) Optimal policy in OG models, *Journal of Economic Theory*, 90, 62-83.

[13] Goenka, A. (1994a) Fiscal rules and extrinsic uncertainty, *Economic Theory*, 4(3), 401-416.

[14] Goenka, A. (1994b) Rationing and sunspot equilibria, *Journal of Economic Theory*, 64(2), 424-442.

[15] Goenka, A. and Liu, L. (2012) Infectious diseases and endogenous fluctuations, *Economic Theory*, 50:125-149.

[16] Goenka, A., Jafarey, S. and Pouliot, W. (2012) Pollution, mortality and optimal environmental policy, Department of Economics Working Paper 12/07, City University, London.

[17] Grandmont, J.-M. (1985) Endogenous competitive business cycles, *Econometrica* 53, 995-1046.

[18] Grandmont, J.-M. (1986) Stabilizing competitive business cycles, *Journal of Economic Theory* 40, 57-76.

[19] Health Effects Institute (2010) Outdoor air pollution and the developing countries of Asia: A comprehensive review, Special Report No. 18, Cambridge, MA. Health Effects Institute.

[20] Huang, W., *et al.* (2012) Seasonal Variation of Chemical Species Associated with Short-term Mortality Effects of PM2.5 in Xian, A Central City in China, *American Journal of Epidemiology* 175(6): 556-566.

[21] John, A., and Pecchenino, R. (1994) An overlapping generations model of growth and the environment, *The Economic Journal* 104: 1393-1410.

[22] John, A., Pecchenino, Shimmelpfenning, Schreft, S. (1995) Short-lived agents and long-lived environemnt, *Journal of Public Economics* 58: 127-141.

[23] Jouvet, P-A., Pestieau, P., and Ponthiere, G. (2007) Longevity and environmental quality in an OLG model, *Journal of Economics* 100: 191-216.

[24] Mariani, M., Perez-Barahona, A., and Raffin, N. (2010) Life expectancy and the environment, *Journal of Dynamics and Control* 34: 798-815.

[25] Miller K.A., *et al.* (2007) Long-term exposure to air pollution and incidence of cardiovascular events in women, *New England Journal of Medicine*, 356: 447-458.

[26] Palivos, T., and Varvarigos, D. (2010) Pollution abatement as a source of stabilisation and long-run growth, University of Leicester Working Paper No. 11/04.

[27] Pope, A., *et al.* (2004) Cardiovascular mortality and long-term exposure to particulate air pollution: Epidemiological evidence of general pathophysiological pathways of disease, *Circulation* 109: 71-77.

[28] Preston, S. (1975) The changing relation between mortality and level of economic growth. *Population Studies* 29: 231-248.

[29] Raffin, N. and Seegmuller, T. (2014) Longevity, pollution and growth, *Mathematical Social Sciences* 69, Issue C, 22-33.

[30] Samet, JM., Dominici, F., Curriero, F.C., Coursac, I., and Zeger, S.L. (2000) Fine particulate air pollution and mortality in 20 U.S. Cities, 19871994. *New England Journal of Medicine* 343:1742-1749.

[31] Smith, B.D. (1994) Efficiency and determinacy of equilibrium under inflation targeting, *Economic Theory* 4, 327-344.

[32] Stokey, N.L. (1998) Are there limits to growth? *International Economic Review* 39, 1-31.

[33] Varotsos, C., Ondov, J., Efstathiou, M. (2005) Scaling properties of air pollution in Athens, Greece and Baltimore, Maryland, *Atmospheric Environment* 39: 40414047

[34] Varvarigos, D. (2008) Environmental quality, life expectancy, and sustainable economic growth, University of Leicester Working Paper No. 08/19.

[35] Varvarigos, D. (2014) Endogenous longevity and the joint dynamics of pollution and capital accumulation, *Environment and Development Economics* 19, 393-416.

[36] Viegi, G., Maio, S., Pistelli, F., Baldacci, S., and Carrozzi, L. (2006) Epidemiology of chronic obstructive pulmonary disease: Health effects of air pollution, *Respirology* 11, 523-532.

[37] Wang, M., Zhang, J., and Bhattacharya, J. (2015) Optimal health and environmental policies in a pollution-growth nexus, *Journal of Environmental Economics and Management* 71: 160179.

[38] Windsor, H.L. and Tuomi, R. (2001) Scaling and persistence of UK pollution, *Atmospheric Environment* 35: 45454556.

[39] Woodford, M. (1994a) Determinacy of equilibrium under alternative policy regimes, *Economic Theory* 4, 323-326.

[40] Woodford, M. (1994b) Monetary policy and price level determinacy in a cash-in-advance economy, *Economic Theory*, 4, 345-380.

[41] Zeka, A., Zanobetti, A., and Schwartz, J. (2005) Short term effects of particulate matter on cause specific mortality: Effects of lags and modification by city characteristics, *Occupational and Environmental Medicine* 62(10): 718-725.

APPENDICES

A1: Derivation of $\partial^2\pi/\partial\tau\partial k$:

Recall that

$$\frac{\partial\pi}{\partial\tau} = \frac{\pi^A\psi\gamma Ak^\alpha\delta z^{\delta-1}}{(1+z^\delta)^2}$$

Note that we can also write this as

$$\frac{\partial\pi}{\partial\tau} = \frac{\pi\psi\gamma Ak^\alpha\delta z^{\delta-1}}{1+z^\delta}$$

Taking the derivative of the above with respect to k (after some straightforward rearrangement):

$$\frac{\partial^2\pi}{\partial\tau\partial k} = \frac{\alpha}{k}\frac{\partial\pi}{\partial\tau} + \frac{1}{\pi^A}\frac{\partial\pi}{\partial\tau}\frac{\partial\pi^A}{\partial k} + \frac{1}{z(1+z^\delta)}\frac{\partial\pi}{\partial\tau}[(\delta-1)-(\delta+1)z^\delta]\frac{\partial z}{\partial k}$$

where

$$\frac{\partial\pi^A}{\partial k} = \frac{\alpha}{k}\frac{\vartheta(1-\underline{\pi})y^\vartheta}{(1+y^\vartheta)^2} = \frac{\alpha}{k}\frac{\vartheta(1-\underline{\pi})y^\vartheta}{(1+y^\vartheta)}\frac{\pi(1+z^\delta)}{(\underline{\pi}+y^\vartheta))}$$

and

$$\frac{\partial z}{\partial k} = \frac{\alpha\gamma(1-\psi\tau)Ak^\alpha}{k} = \frac{\alpha z}{k}$$

The right hand side of the main derivative can be written as

$$\frac{\partial\pi}{\partial\tau}\left[\frac{\alpha}{k} + \frac{(1+z^\delta)}{\pi^A}\frac{\alpha\pi}{k}\frac{\vartheta(1-\underline{\pi})y^\vartheta}{(1+y^\vartheta)(\underline{\pi}+y^\vartheta)} + \frac{\alpha}{k}\frac{z-\phi z'}{z(1+z^\delta)}[(\delta-1)-(\delta+1)z^\delta]\right]$$

Finally, expanding the term in square brackets involving z^δ and noting the definition of π, we get

$$\frac{\partial\pi}{\partial\tau}\left[\frac{\alpha}{k} + \frac{1}{\pi}\left\{\frac{\alpha\pi}{k}\left(\frac{\vartheta(1-\underline{\pi})y^\vartheta}{(1+y^\vartheta)(\underline{\pi}+y^\vartheta)} - \frac{z\delta z^{\delta-1}}{(1+z^\delta)}\right) + \frac{\alpha\pi\delta}{k}\frac{z}{(1+z^\delta)z} - \frac{\alpha\pi}{k}\frac{z}{z}\right\}\right];$$

from which, noting the definition of $\partial\pi/\partial k$, it follows that

$$\frac{\partial^2\pi}{\partial\tau\partial k} = \frac{\partial\pi}{\partial\tau}\frac{1}{k}\left[\alpha + \frac{k}{\pi}\frac{\partial\pi}{\partial k} + \frac{\alpha\delta z}{(1+z^\delta)z} - \alpha\right];$$

leading to the desired result.

A2: An example of $\pi(k)$:

Assuming the functional form:

$$\pi = \pi^A \pi^B$$

where

$$\pi^A = \frac{\underline{\pi} + y^\vartheta}{1 + y^\vartheta}$$

then it can be shown that $\pi_y^A > 0$ if $\underline{\pi} < 1$ and that $\pi_{yy}^A \leq 0$ if and only if $y \leq [(\vartheta - 1)/(1 + \vartheta)]^{1/\vartheta}$ so that for any $\vartheta > 1$, $\pi^A(y)$ is S-shaped upwards.

If similarly,

$$\pi^B = \frac{1}{1 + z^\delta}$$

then it can be shown that $\pi^B < 0$ and that $\pi_{zz}^B \leq 0$ if and only if $z \leq [(\delta - 1)/(1 + \delta)]^{1/\delta}$ so that for any $\delta > 1$, $\pi^B(z)$ is reverse S-shaped downwards.

Thus, the above function satisfies the sufficient conditions for multiple steady states, and after imposing the steady state relationship between y, z and k and totally differentiating, that a sufficient condition for $\pi'(k)$ to satisfy the conditions of Lemma 1 as k approaches zero is

$$\min\{\vartheta, \delta\} > \frac{1}{\alpha} > 1.$$

This ensures $\lim_{k \to 0} \pi'(k) = 0$, which is stronger than what is needed for Lemma 1.

If we consider a special case where $\underline{\pi} = 0$, then $\pi'(0) = 0$ so long as $\vartheta > 1/\alpha$. For this case, it can be shown that a weaker condition

$$\vartheta > \frac{1 - \alpha}{\alpha}$$

suffices to generate $\mathtt{G}'(0) = 0$.

This is because the combination of the terms

$$\mathtt{G}'(k) = \left[\frac{\Gamma k^\alpha}{1 + \pi(k)}\right] \left[\alpha \frac{\pi(k)}{k} - \frac{\pi'(k)}{1 + \pi(k)}\right].$$

can converge to zero even if each term inside the square brackets does not.

Another special case is to assume $\pi^A = \bar{\pi}$ so that growth affects survival probability only through pollution. This case can also lead to multiple steady states if $\delta > 1/\alpha$ and can also be used for studying optimal tax policy, but because it implies a counter-factually monotonic and negative impact of growth on survival, we do not pursue it.